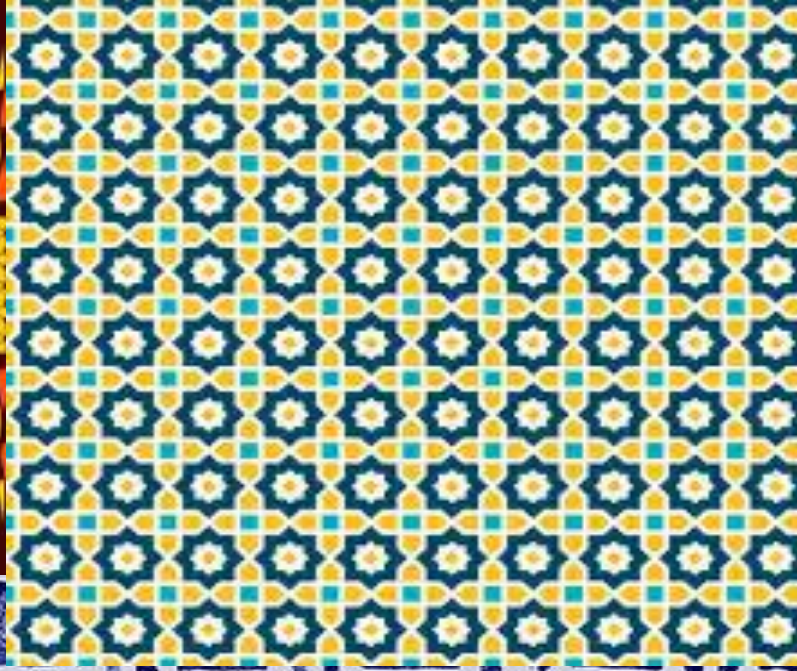

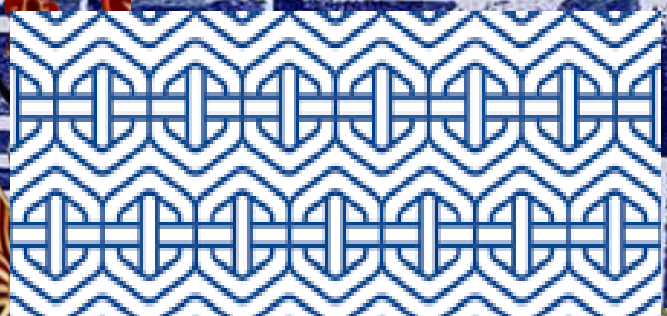

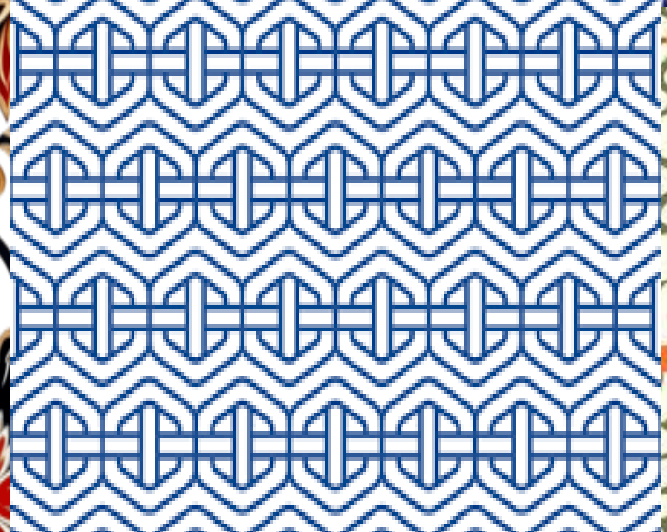

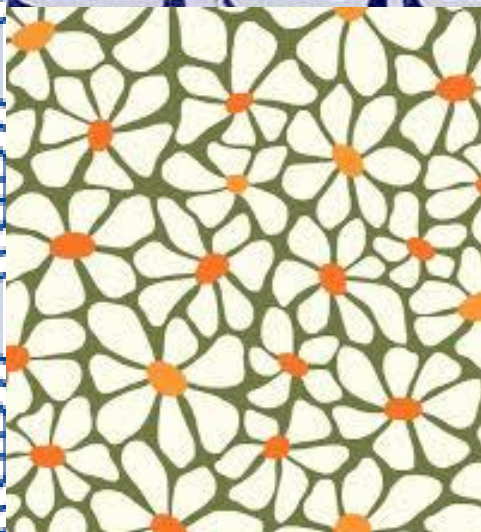

General Topology

In progress ...

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The word *topology was coined in the XIX century.* It *comes from the Greek* $\boldsymbol{\tau o \pi o} =$ place, position , and $\boldsymbol{\lambda o \gamma o \varsigma} =$ science, study... which may mean *the study of position* . The word was invented by German mathematician Johann **Listing**$(1808-1882)$ in 1847 in his book *Vorstudien zur Topologie*. Topology as a mathematical subject is seen as a part of modern geometry and it is a very abstract study of continuity and the related notions : proximity, limit of functions , connectivity,…

As usual, any word can change of meaning according to the context, it is likewise for the word *topology* , it has been given many significations :topology(science), topology of a space, topological propriety, topological space,…. *General topology* (called also *point-set topology*) is often linked to algebraic topology (also called *combinatorial topology*) , for example the book *History of Topology* (1999) edited by English mathematician Ioan **James**$(1928-2025)$ treats mainly of the history of algebraic topology. We read in **Moore** $(p.9)$:

In the mathematical literature the term point-set topology is ambiguous. Within this book it refers to the theory, sometimes called Mannigfaltigkeitslehre or the theory of point sets, that ***Cantor*** *originated and that utilized such notions as limit point, closed set, and derived set in* $\mathbb{R}^n$*. Largely through the book of* ***Hausdaurff****, this theory later grew into general topology which considered more general spaces.*

I am interested here of *general topology* that studies and generalizes the proprieties $\mathbb{R}$ (and general $\mathbb{R}^n$) to abstract sets. It is the study of the geometry of points of a set, for we work with sets of points : set of functions, set des points of geometry…. It studies proprieties called *topological* , also maps named homeomorphisms, those functions that leave some algebraic proprieties intact as **Dixmier**

Classify topological spaces

General topology has not become only an introduction to algebraic topology, but it has become a language of modern mathematics as **Jameson** said:

It seems to be sufficiently realized that fully-blooded General Topology is essentially the servant of Functional Analysis, and not of Algebraic Topology.

General topology is really an introduction to different geometries : algebraic topology, differential geometry, and to functional analysis …. Many Initiated topology ideas German Bernhard **Riemann**

$(1826-1866)$ and **Cantor** then French **Fréchet** $(18-19)$ a la topology. The concept of the continuity of functions. by **Cantor** $(18-191)$, **Borel** (18-19), **Jordan** $(1838-1922)$, **Lebesgue** $(1875-1941)$…to understand the set of real numbers and $\mathbb{R}$ the first topological space.

Topological studies abound, so we talk in this text of the necessary. I try here to give as much results as possible and exercises for training. The topology that interest us here is general, I want to be for all students, can be an introduction as to functional analysis, measure theory, algebraic topology.

Topological Spaces

Sets :.....5
Neighborhoods:.....1
Topology on a Set:.....1
Remarkable Points:.....1
Compactness:.....1
Continuity:.....1
Remarkable Topologies:.....1
k :.....1

Metric Spaces

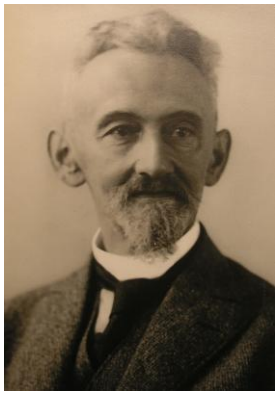
Distance :.....
metric space:.....
Balls :.....
continuity :.....
Groups :.....

Normed Spaces

Norms :.....
normed space:.....
normed Vector Spaces :.....
Banach spaces:.....
Hilbert spaces:.....

Topological Spaces

◯ Sets:



Hausdaurf

A set E is a collection of objects defined by a certain rule. So the nature of these objects is a problem. We have to be careful when using the natural language in defining sets. We will see for example that *the set of all sets* cannot exist because it is too big. Another more disturbing set is : *The set of all sets that do not belong to themselves*, which leads to the flagrante contradiction[1]

$$E \in E \iff E \notin E$$

Mathematicians at the beginning of the XX century made great efforts to get rid of these inconsistencies by building a new theory in what is called *Axiomatic Set Theory*[2].

So we say that a *set* is a collection of elements(objects) and we can define what a set E is if we can say that an element x is in it or not, so we talk of a *well-defined* set.. If an element x belongs to E we write

$$x \in E$$

If a set A is in another set E we name A a *subset* of E, we note this relation by $A \subset E$. If not we write $A \not\subset E$. Any set E is a subset of E. We note $A \subseteq E$ if it is possible that A equals to E and $A \subsetneq E$ if A is a proper subset of E. The set of all subsets of a set E is called the *power set* and noted $\mathcal{P}(E)$. To the elements of a set, we use the *braces* { } . When $E = \{a, b\}$ we have

$$\mathcal{P}(E) = \{E, \emptyset, \{a\}, \{b\}\}$$

The within braces is called the : *roster notation* . another is *set builder*

---

[1] Belonging to this subject,the so called self−reference sentences like the Liar Barber.
[2] See for example the book of Daniel **Cunningham** : *Set Theory: A First Course*.

$$E = \{x : x = 2, \}$$

The inclusion $\subset$ is different from $\in$. For inclusion we can write

$$E \subset E \subset E \subset E \ldots$$

but we cannot write $x \notin x$, also we write $x \in E$ but $x \notin \mathcal{P}(E)$. both $x \in E$ and $E \in F$ are true , sometimes we consider $E \in F$ but not $E \subset F$, we have $A \subset E \Leftrightarrow A \in \mathcal{P}(E)$.

Two sets E and F are equals if they have the same elements. We say that $E = F$ if $E \subset F$ and $F \subset E$. It follows that the sets $\{3, 3\}$ and $\{3\}$ are the same (think of solution of an algebraic equation).

There is a set called the *empty set* defined as

$$\emptyset = \{x : x \neq x\}$$

We have $\emptyset \subset E$ as $E = E \cup \emptyset$. we can define by a infinitely many methods.

In we need a *universal set*(not the sets of all sets!!!) to the complementary. Let $A \subset E$ we say that $E = A \cup A^c$, where A^c is complementary of A in E. This leads to the definition

$$A^c = E \backslash A = E - A = \{x : x \in E \; and \; x \notin A\}$$

some proprieties of the complementary of sets is $(A^c)^c = A$ and the Morgan law

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

Another relation between sets is the *symmetric difference*

$$A \Delta B = \{x : x \in A \; and \; x \notin B\}$$

we have : $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

Maps

A function(or a map) is a relation between two sets E and F

$$f : E \longrightarrow F$$
$$x \longmapsto f(x)$$

f is the function(the law) and $f(x)$ is the value of f at x. We can see a function is a subset in the Cartesian product of two sets $E \times F$. Note that $E \times F \neq F \times E$ in general.

The family of all functions from E to F is noted F^E and it is a subset of the Cartesian product $E \times F$. A *graph* of a map f is the subset of $E \times F$.
$f: E \longrightarrow F$ is *injective* (onto) if $f(x_1) = f(x_2) \implies x_1 = x_2$. If $E \subset F$ the map is an *injection* and is noted $E \hookrightarrow F$.

f is *surjective* (onto) if $\forall y \in F \quad \exists x \in E : f(x) = y$. If f is both injective and we *bijective*(one to one).

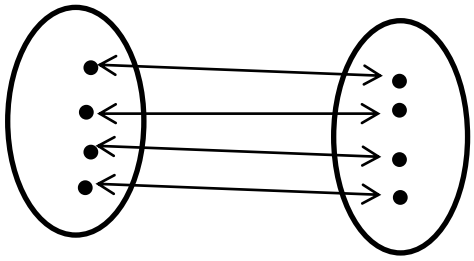


Fig. bijection between two sets

If f is bijective then $f^{-1} \circ f = id_E$ and $f \circ f^{-1} = id_F$. if : $E \longrightarrow E$. a subset of the is $id_E: E \longrightarrow E$

Even if f^{-1} does not exist, the inverse image $f^{-1}(A)$ always exists. We have another time $\emptyset \subset E$ as $f^{-1}(x) \subset E$. Some general results: For a set of indices in I, we have

$$f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_i)$$

but

$$f\left(\bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} f(A_i)$$

We have the equality $f\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f(A_i)$ if f is injective. The result is true for any infinite sum. counterexample:

Cardinal

Sets can be finite or infinite. If E is finite the number if its elements is called its *cardinal* and noted $\sharp\, E$, $|E|$ or $Card(E)$. If a set is finite, its cardinal is finite and we can write $Card(E) = n$. Two sets are *equipotent* if they have the same cardinal. Two equipotent finite sets have the same number of elements, two equipotent infinite sets we can find a bijection.

Bernstein theorem[3]

If there exists an injection $E \longrightarrow F$ and an injection $F \longrightarrow E$ then there is a bijection between E and F.

The idea is two find two injections is standard : if $Card(E) \leq Card(F)$ and $Card(F) \leq Card(E)$ then $Card(E) = Card(F)$.

Infinite sets are not new, **Euclid** proved that the set of primes $\mathbb{P}$ is not finite. The trouble with infinite sets is an infinite set can be *equipotent* to one of its subsets this exactly the definition

Definition

A set E is infinite if there a bijection between E and $A \subset E$.

In the case of $\mathcal{P}(E)$ we have

$$Card\left(\mathcal{P}(E)\right) = 2^{Card(E)}$$

If E is infinite the former *number* is nonsense. But infinities can be generalized to *infinite numbers* and the man has who responsible of the creation of these subtleties was German Georg **Cantor**$(18 - 191)$, at first he was not understood by his fellow mathematicians but now it is established the if fruitful for mathematics.

The system begin with the set $\mathbb{N}$. We say that a set E is *countable* if there is a bijection between E and a subset of $\mathbb{N}$. So any finite set is countable.

$$Card(\mathbb{N}) = \aleph_0$$

[3] By **Cantor** in 18.

The Hebrew symbol $\aleph_0$ is posed by **Cantor**. There is a bijection between E and a subset of $\mathbb{N}$ then E is *non countable*. Countability is vital for mathematics, since all sequences u_n are summed over $\mathbb{N}$

$$\sum_{n\in\mathbb{N}} u_n$$

but when E is a non−countable set we work with *familles of indices* in a set I . for example the sum

$$\bigcup_{i\in I} u_i$$

where I can be countable or not.

The sets $\mathbb{Z}$, $\mathbb{P}$ (set of prime numbers) are countable sets, and

$$Card(\mathbb{N}) = Card(\mathbb{Z}) = Card(\mathbb{P}) = \aleph_0$$

Theorem 1

The cardinal of finite number of countable is coble is $\aleph_0$.

Cantor could prove an amazing thing, the fact that $Card(\mathbb{Q}) = \aleph_0$! Although there are infinitely many rational numbers between every two integers $(n, n+1)$, there is a bijection between it and $\mathbb{N}$!! so

$$Card(\mathbb{N}) = Card(\mathbb{Z}) = Card(\mathbb{P}) = Card(\mathbb{Q}) = \aleph_0$$

Another fact proved by **Cantor** is that $\mathbb{R}$ is different from $\mathbb{N}$, namely there is not bijection between $\mathbb{R}$ and $\mathbb{N}$, there is even no bijection between $\mathbb{R}$ and $\mathbb{Q}$. So $\mathbb{R}$ has a another cardinal That is the continuum cardinal noted : $\mathfrak{c}$ so

$$Card(\mathbb{R}) = \mathfrak{c}$$

From theorem 1 even $\mathbb{I}$ the set of irrational numbers, is not countable so

$$Card(\mathbb{R}) = Card(\mathbb{I}) = \mathfrak{c}$$

As for finite numbers, the infinite cardinals are not the same, in fact there are an infinity of cardinals. This follows from the inequity

Theorem

Let E be a set. Then

$$2^{Card(E)} > Card(E)$$

This result was proved by **Cantor** in 1891. We return to the expression : "*The set of all sets*", we have said that this set cannot exist because the above theorem implies

$$2^{Card(E)} > Card(E)$$

a contradiction.

◯ *Neighborhoods:*

The subject of general topology is to observe *open* intervals $]a, b[$ to abstract sets. Its aim is to generalize important proprieties of $\mathbb{R}$ to very general sets. First we begin by defining a general interval in $\mathbb{R}$.

Definition:

A set $I \in \mathbb{R}$ is an *interval* if : $\forall x, y \in I$ and $x < z < y$ then $z \in I$.

Intervals can be open $]a, b[$, closed $[a, b]$ or neither $]a, b]$. An open interval $I =]a, b[$ in $\mathbb{R}$ has an interesting propriety

$$\forall \varepsilon > 0 \ : \ ]a + \varepsilon, b - \varepsilon[\ \subset \ ]a, b[.$$

which is translated by saying that an open interval is the *neighborhood* of all its points. to say that I is open means

$$\forall \, x \in I, \exists \,]\varepsilon_1, \varepsilon_2[\ : \ x \in]\varepsilon_1, \varepsilon_2[\subset \ I.$$

The set $\mathbb{R} =]-\infty, +\infty[$ is open . $\emptyset$ is also open because according to the definition of openness is that x ther interval but there is no $x \in \emptyset$ in that does not verify the propriety $\left(P = \bar{\bar{P}}\right)$ which. We can also in $\mathbb{R}^2$. So $\boldsymbol{D}$ in is a neighborhood of a point x if there exits an open region $\mathcal{O}$ in $\mathbb{R}^2$ such that $x \subset \mathcal{O} \subset \boldsymbol{D}$. The same example is in the plan that we can as $\mathbb{R}^2$. D id open

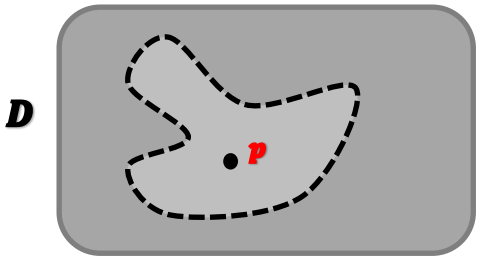


Fig. $\boldsymbol{D}$ is a neighborhood of p

For not neighborhood

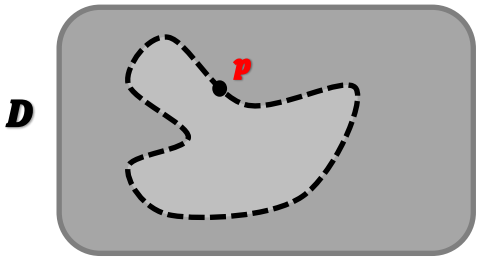


Fig. $\boldsymbol{D}$ is not a neighborhood of p

For not neighborhood

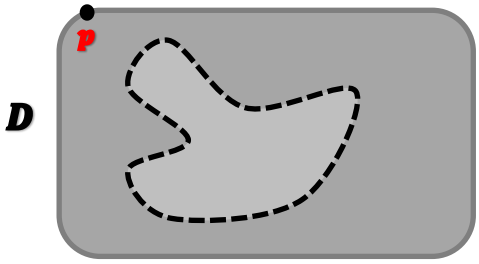


Fig. $\boldsymbol{D}$ is not a neighborhood of p

The same is true for $\mathbb{R}^3$. We can generalize for $\mathbb{R}^n$ when $n > 3$. We can from the properties of interval in Infer the definition of a topological space τ in the next section. to satisfy , we will usingdisks(balls) of radius $\varepsilon > 0$ because they are one size.

◯ *Topology on a Set:*

The set $\mathbb{R}$ is a very important set in mathematics, of a very complicated structure that has been analyzed in scrutiny by mathematicians since the XIX century. From what we have seen before, we can remark that the open intervals I in $\mathbb{R}$ have three proprieties :

– $\mathbb{R}$ and $\emptyset$ are open.
– $\bigcup_{i=1}^{+\infty} I_i$ is open.
– $\bigcap_{i=1}^{n} I_i$ is open.

Topology study how the points of an abstract set are situated inside the set. These new sets like open intervals called *open sets*. The study of these open sets has become non instinctive at all. It is the birth of the stream of abstraction in mathematics which revolutionized the ideas of algebra, geometry,…

The set $\mathbb{R}$ is composed of real numbers, $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$, it is the *continuum*.

A *space* in mathematics is a non-empty set with a structure : algebraic, topological, metric, differential,…A topological space is the most abstract set where we can define neighborhoods ,…this will helps us to define “continuity”, “closeness”,… General (or point set) topology is a part of algebraic topology and has become an introduction to many of mathematics subject as real analysis and function analysis as it deals with the topological proprieties of the $\mathbb{R}^n$. We can from the properties of interval in Infer the definition of a topological space τ . Let E be a non-empty set, with subsets τ a set of part of E $(\tau \subset \mathcal{P}(E)$ such that

Definition:

Let E be a set, we say that $\boldsymbol{\tau}$ is a *topology* over E if

$-\ E$ and $\emptyset \in \boldsymbol{\tau}$
$-\ A_1, \dots, A_n \in \tau$ then : $\bigcap_{i=1}^{n} A_i \in \boldsymbol{\tau}$
$-\ \bigcup_{i\in I} A_i \in \boldsymbol{\tau}$

These axioms were posed by German Felix **Hausdaurf**$(1868-1942)$ around 1914. We say that $\boldsymbol{\tau}$ is a *topology over* E. The couple $(\mathrm{E}, \boldsymbol{\tau})$, or merely E, is called *topological space*. When we define a topology over, E itself is a topological space. The elements of a topological space are the *points* (geometry). A topology $\boldsymbol{\tau}$ is the set of opens. We say also that (E, τ) or simply E is a *topological space* .

The sets A_i define a topology on E we note them $\emptyset_i$.The set $\emptyset$ is open. A subset A is open : $\forall x \in A, \mathcal{O}: x \in \mathcal{O} \subset A$. Openness comes from limit of variables in analysis. in the *usual* topology $\mathbb{R}$ is open $\mathbb{R} =]+\infty, -\infty[$.

Remarks:

–Similarly as in $\mathbb{R}$, we say that A is a neighborhood of x if there exists an open $x \in O \subset A$. It is evident that for every topology τ over E we have $\tau \subset \mathcal{P}(E)$.

–One must distinguish between an open set of real numbers $]a, b[\cup]c, d[$ and an open interval $]a, b[$ ⏏

Examples:
– We can indicate the biggest topology and smallest topology on a set E. The first is the *discrete* topology, it is just simply $\tau_d = \mathcal{P}(E)$, it contains all the singletons. The second is $\tau_i = \{E, \emptyset\}$ called the *indiscrete* topology.

– More the space has opens more its topology is richer. For a set E, la topologie $\mathcal{P}$(E) is the richer and $\{\mathrm{E}, \emptyset\}$ is the poorest. Since $\{\mathrm{E}, \emptyset\} \subset \mathcal{P}(\mathrm{E})$, we say that $\mathcal{P}(E)$ is *finer* than $\{\mathrm{E}, \emptyset\}$, $\{E, \emptyset\}$is *coercer*. is the *coarsest* topology on a set E. However we cannot always compare between two topologies $\boldsymbol{\tau_1}$ et $\boldsymbol{\tau_2}$, for $E = \{\mathrm{a}, b\}$, we cannot compare between $\boldsymbol{\tau_1} = \{\mathrm{a}, \mathrm{E}, \emptyset\}$ and $\boldsymbol{\tau_2} = \{b, \mathrm{E}, \emptyset\}$ ⏏

Closed Sets:

A *closed* interval is one of the form $[a, b]$. A set $A \subset E$ is closed if it is the complementary A^c or $E - A$ is open in E. A set can be both open and closed In a discrete topology τ_d, every set is open and closed. In any topology, E and $\emptyset$ are always open and closed. It turns out that we can define a topology τ in terms of closed subsets.

Definition

$i)$ $E, \emptyset \in \tau$ are closed .
$ii)$ if $A_1, \dots, A_n$ are closed sets then $\bigcup_1^n A_i \in \tau$ is closed.
$iii)$ if $A_{i\in I}$ then $\bigcap_{i\in I} A_i \in \tau$ is closed.

Remarks:

–The intersection $\bigcap A_i$ is finite for we have a counter–exemple : $\bigcap_{n=1}^{\infty}]-\frac{1}{n}, \frac{1}{n}[= \{0\}$ which is not open in $\mathbb{R}$. A closed set A by definition is the complementary of an open .

$$F = A^c$$

– let E be a set and a topology over E. A set A can be open in a topology τ_1 and not in another τ_2, here we real topology $(\mathbb{R})$. By similarity as $]a, b]$ is neither open nor closed there subset that neither open nor closed ⏏

Examples:

– $\mathbb{R}$ is a topological space noted $(\mathbb{R}, \boldsymbol{\tau})$, the topology $\boldsymbol{\tau}$ is called *usual* of the opens intervals :]a, b[ ]a , b[∪]c, d[, $]-\infty, +\infty[= \bigcup_{n=0}^{n=+\infty}]-n, n[$

There are in E parts which are neither des open nor closed , as in $\mathbb{R}$ les intervalles semi ouvert [a, b[. which makes the difference between closed and open is their topologies are different,

–a closed too is not a neighborhood of all its points. But a closed has also its advantages :

–Every function on a closed interval reaches its greatest value and smallest value, which is not true for an open.

– Every continuous function f on a bounded closed interval is uniformly continuous, which is not true for an open ⏏

We are going now to elucidate the concept of proximity, nearness ….in a topological space. The notion of nearness is very vague, even in an interval $[a, b]$ there is an infinity of points between any two point a_i , b_i.

Neighborhoods:

The interval $]0,1[$ is the neighborhood of all its points, but $[-1,2[$ is not a neighborhood of all its points, 1 for there is no open interval $\boldsymbol{I}$ such that $-1\in \boldsymbol{I}$ et $\boldsymbol{I}\subset[-1,2[$.

Definition:

$i)$ E is a neighborhood of x.

$ii)$ if $A_1,\dots,A_n$ are neighborhood of x then $\bigcap$ A_i neighborhood of x.

$iii)$ if $A_1,\dots,A_n$ are neighborhood then $\bigcup$ A are neighborhood of x

If $a \in E$, E is a deleted neighborhood of a.

We say that $A \subset E$ is the *neighborhood* of x if there is an open set $\mathcal{O}$ such that $x \in \mathcal{O} \subset A$.

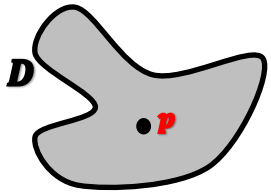


Fig. $\boldsymbol{D}$ is a neighborhood of p

So D is closed

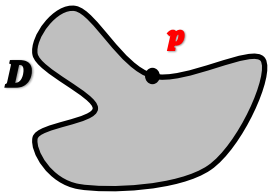


Fig. $\boldsymbol{D}$ is not a neighborhood for p

Remark :

–The One of the earliest books is due to German mathematician Felix **Hausdorff** $(18-19)$ (1914) . afterwards, the concept of topological space was cleared by **Alexandrov** and **Uryshon**(1922) ⏏

the subsets are *open*. An open set(as an open interval $]a,b[$) is the neighborhood of all its elements.

The more open sets two points both belong to, the closer they are. $]a,b[$ is an open set and $[a,b]$ a closed whereas $[a,b[$ is neither open nor closed. what is a closed set ?

Fundamental system of neighborhood:

A neighborhood of a point x can open cosed or neither. We see now another special of neighborhood .

Definition

E si and xE we say that is a fundamental system for x

$$I \subset E$$

Remarks:

–In a it is ⏏

Examples:

– The spaces $\mathbb{R}^n$ are ⏏

Basis:

In a topological space we can have a basis that every subset of the topology, we can it as a union of the basis.

Definition

Let E be a topological space and $\mathcal{B}$ a collection of open sets . $\mathcal{B}$ is a *basis* for the topology τ if every open sets in topology is a union of sets from $\mathcal{B}$.

Every discrete topology τ_d has a basis they are the *singletons*. If the basis is countable we say that it is *second countable*. A space is *first countable* at a point if there is a countable local basis at the point. Every second countable space is first countable, but the inverse is not true.

Remarks:

–In a it is seconde countable

–the basis for a topological space may not be unique ⏏

Examples:

– The intervals form a basis for the usual topology of $\mathbb{R}$

−spaces $\mathbb{R}^n$ are second countable for their usual topology.
− $\tau = \mathcal{P}$ d ⏏

◯ *Remarkable Points:*

In a topological space $\mathbb{R}$ we can define many special points. Similarly any topological space degage points having proprieties that will be of a lot of interest in the sequel:

1) limit Point

Also of accumulation point: Let $(E, \boldsymbol{\tau})$ a topological space, we say that x_0 is a *limit point* of E if every neighborhood of x_0 in E contains a point different of x_0. Or in language :

Definition:

Let $(E, \boldsymbol{\tau})$ a topological space, we say that x_0 is a *limit point* of E if

$$\forall v \in V(a), v \cap E \neq \emptyset$$

the set of limit points of a set E is called *derived set* and noted E'. It was defined by **Cantor** in 1872 in his article.

Remarks:

− Certain authors mean by point limit and accumulation point an infinity of points.
−thee limit point of a set does not necessarily belong to E. ⏏

Examples:

−for $E = [0,1]$; the point 1 is a limit point for E because ⏏

Let [a, b] un point x est d'accumulation s il n'est pas *isolé*

$$\forall v \in V(a), v \cap A = \{a\}$$

for $E = \{-1\} \cup [0,1]$; the point -1 is not a limit point for A.

Proposition:

A set E is closed iff it contains all its limit points.

Proof

We can be this prove that $[a, +\infty[$ is closed

On va voir que le fermé a beaucoup dans la théorie des équations différentielles. On a aussi

Bolzano –Weierstrass theorem :

Every infinite and bounded set E, has at least a limit point .

this result on the set of real numbers $\mathbb{R}$ has been known before the birth of topology and its abstract ideas. I was emitted by Bohemian mathematician Bernard **Bolzano**$(1781 - 1848)$ for the set of real numbers (topological space $\mathbb{R}$!) then proved by German **Weierstrass**$(1815 - 1895)$.

Fig. uuuuu

Il belongs to the topological proprieties of $\mathbb{R}$. It has equivalent statements as we will see in the subsequent chapters.

Interior point :

the interior of set : nor closed

Definition :

Let $(E, \boldsymbol{\tau})$, a topological space , we say that x is an ***interior*** point of E if E is a neighborhood of x

$$\exists v \in V(x) \ such\ that : x \in v \ \ and \ \ v \subset E$$

We the interior by the symbol $Int(E)$

Remarks:

–Any open set is equal to its interior. We have always $E^0 \subset E$ but no necessarily $E \subset E^0$.

– The interior of a set E is the biggest open set contained in E.
– It permits to distinguish what is interior from that exterior. If $x_0 \notin E^o$ we say that x_0 belongs to the exterior of E ⏏

Examples:

– For $E = \mathbb{R}$ and $A = [0,1]$ we have $A^o =]0,1[$. we notice that $A^o \subset A$ but $A \not\subset A^o$
– For $E = \mathbb{R}$ and $A = \mathbb{Z}$ we have $\mathbb{Z}^o = \emptyset$. Because suppose $x \in Int(\mathbb{Z})$ then there exists an open $\mathcal{O}$ such that but there no such $\mathcal{O} \subset \mathbb{Z}$ ⏏

Adherent point:

There are between three points : limit, adherent and

Definition :

let E be a topological space, we say that x is an ***adherence*** point of A si tout voisinage de x dans E contient un point de A.

The adherence of a set E noted $\overline{\mathrm{E}}$, c'est l'ensemble de tous ses points adhérents , elle aussi appelée *fermeture*. $\overline{\mathrm{E}}$ is the smallest set that contains E. We have his result if E is closed then

$$\overline{\mathrm{E}} = E$$

in $\mathbb{R}$, the adherent point of the interval $[0,1[$ is the closed interval $[0,1]$. It is a the most small closed containing E. le point d'adhérence est très lié aux suites et leur convergence dans E. Let u_n be a sequance de dans E si u_n converges ins E alors l'ensemble infini des valeurs u_n ont un point d'adhérence qui est la limite de u_n.

3) frontier point :

Definition:

Let E be a set. We say that x est un point ***frontier*** of A si tout voisinage de x dans E contient un point de A.

On note la frontière de A par $Fr(A)$ ou ∂A et on a

$$\partial A = \partial(A^c)$$

also

$$\partial A = \overline{A} \backslash A^{O}$$

If the boundary of E is empty ($\partial E = \emptyset$) then E is closed.

here is an open set $\mathcal{O}$ such that $x \in \mathcal{O} \subset A$.

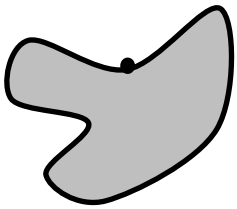


Fig. point on the boundary of $\boldsymbol{D}$.

Examples:

–for $A = [-1,1]$ we have

i) $\overline{\mathrm{A}} = [-1,1]$
ii) $Fr(A) = \{-1,1\}$
iii) $A^{O} =]-1,1[$ ⏏

we have the relation between closure and limit points

$$\bar{E} = E \cup E'$$

Density and filter :

From the closure we have if is clos then $\overline{\mathrm{E}} = F$ we say then that E is *dense* in F. This has with the set of real number and rational. The set of rational number is dense in, for every there a ration between them we have seen that we will use the close closure $\overline{\mathrm{E}}$

an intuitive idea known between the two sets $\mathbb{Q}$ and $\mathbb{R}$ set que between two rational there exists always irrational numbers.

$$\forall a < b \in \mathbb{Q}\,, \exists x \in \mathbb{R} : \; a < x < b.$$

also the irrational $\mathbb{I}$ is dense

$$\forall a < b \in \mathbb{I}\,, \exists x \in \mathbb{R} : \; a < x < b.$$

we have very important results, on dit que E est dense dans F si et on écrit

$$\overline{\mathrm{E}} = F$$

What make difference between $\mathbb{R}$ and $\mathbb{Q}$ is greatest lower bound and . we have density by the fundamental $\overline{\mathbb{Q}} = \mathbb{R}$. a similar about density is Stone−Weierstrass theorem. It aws first proved by **Weierstrass** then by American Marshall **Stone**$(1903 - 1989)$. Que la ensemble des polynômes est dense dans l'ensemble des fonctions continues comme on peut un nombre rre par des rationnels par le dégrée que l'on veut on peut aussi approximer une fonction continue quelconque par un polynôme. We have a latter in chapter :

Theorem:

let E if x then $\exists x, \ldots x \in E$ such that

$$x \in \overline{\mathrm{E}} \iff \exists x, \ldots x \in E : x_n \longrightarrow x :$$

In a general topological space, there is no concept of sequence, a concept which necessitates a *distance* that we have not yet defined. So to this deficiency and in 1937 French mathematician Henri **Cartan** $(1904 - 2008)$ introduced the abstract concept of *filter*.

Definition :

let E be a set, we say that a family $\mathcal{F}$ of parties of E is a *filter over E*

$-\emptyset \notin \mathcal{F}$

$-si\ A \in \mathcal{F}\ \ et\ A \subset B\ alors\quad B \in \mathcal{F}$
$-si\ \ A_1, A_2 \in \mathcal{F}\ alors\ A_1 \cap A_2 \in \mathcal{F}$

We say that u_n converges towards α *following* the filter $\mathcal{F}$

◯ *Compactness:*

To define the concept of compactness we have to some preliminaries. We say that a set C is a *cover* for a set E if $E \subset C$. When C is open we say *open cover*. For example $[0,1]$ has the cover $\left[0,\frac{1}{2}\right] \cup \left]\frac{1}{2},1\right]$ as $[0,1] \subset [0,\frac{1}{2}] \cup]\frac{1}{2},1]$.

First the the Lebesgue—Borel propriety

Definition:

a space X is *compact* if from every cover of X we can extract a finite subcover.

For example $]0,1]$ has the cover it with intervals of the form $]\frac{1}{n},1]$ such that

$$]0,1] = \bigcup_{n=1}^{+\infty}]\frac{1}{n},1]$$

we cannot extract a finite subcover. The set of all open balls of radius $\varepsilon > 0$ is an open cover of the plane $\mathbb{R}^2$.

Examples:

—$\mathbb{R}$ is not compact, but $\mathbb{R} \cup \infty = \overline{\mathbb{R}}$ is compact.
—The sphere S^2 is compact, it is closed and bounded in $\mathbb{R}^3$.

⏏
if $X = \mathbb{R}^n$ we have the Heine— Borel theorem

Heine— Borel theorem:

in a is compact iff is bounded and closed.

Examples:

— the compact sets in $\mathbb{R}$ are the interval $[a,b]$ ⏏

A space can be not compact but we some and locally.

Definition:

a space is *locally compact* every has a compact neighborhood.

Remarks:

—$\mathbb{R}$ is *locally compact* because every $x \in \mathbb{R}$ has a compact neighborhood ⏏

Examples:

−$\mathbb{R}$ is *locally compact* because every $x \in \mathbb{R}$ has a compact neighborhood ⏏

Another concept in topology related to compactness is paracompactness due to French Jean **Diuedonné**$(1906 - 1992)$ in his paper " *Une généralisation des espaces compacts*"(1944).

Definition:
a paracompact every

Every compact space is paracompact.

◯ *Remarkable Topologies:*

Cofinite topology:

separate every two

Separated topology:

We say that if a topology τ is *separated*

Definition
if $\exists V_x, V_y$, $V_x \cap V_y = \emptyset$. Every two points are separated by two distinct neighborhoods.

which means that the topology has a sufficiently opens to separate[4] its points. the propriety de separation is very used. We have this result : if E is separated , the limit of a convergent sequence is unique. We can separate every two point x, y by open subsets.

Definition :
we say that $(E, \boldsymbol{\tau})$ is *Hausdaurff* if it contains a dense countable basis .

[4] The **Hausudaurf** in French is *séparé* the word separable space is rare in English.

the idea off Hausdaurff[5] space is because the pathologic cases as of two *planes* that shares the point$(0,0)$ only.

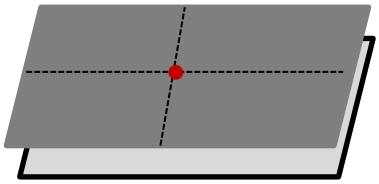


Fig. the point 0 . so the space is not separated

O is not separated⏏

Remarks:

– In a Hausduarf space, the limit of a convergent sequence is unique.

–(or Hausdaurf or T_2)

Examples:

– *pathologic* spaces. Two planes by the 0. So every

– $\tau = \mathcal{P}(E)$ cannot be Hausdaurf.

–there other like T_0

–Let $E = \{a, b\}$ and the topology $\tau = \{\emptyset, E, \{a\}\}$, it is Sierpinski space, an example of a space that is T_0 but not T_1 .

$$T_0 \subset T_1 \subset T_2 \subset T_3 \ldots$$

the same for $\tau = \{\emptyset, E, \{b\}\}$ ⏏

Normal Topology:

Any

Definition
A topological space X is normal if

[5] a Hausdaurf space is also called *separated*, we will not his her the reader should make the difference between *separated and separable*.

Product Topology:

We usually consider product of rel set : $\mathbb{R} \times \ldots \times \mathbb{R}$ whci we note $\mathbb{R}^n$ say this leads us to the *product of topological space.*

f

Definition

A topological space $X \times X$.

and the previous is *finite* product.

Examples :

–cylinder is seen as a product of two topological space $C = S^1 \times I$ ⏏

it is the Tychonov product.

Definition

A topological space $X \times X$. the product of compact topology is compact.

Another is an *infinite* product, Due to Russian Andrey **Tychonov**$(1906 - 1993)$ and is equivalent to the axiom of choice.

Definition

A topological space $X \times X$. the product of compact topology is compact.

◯ ***Quotient topology****:*

when we nous d'une espace topologique dune topologie sur un ensemble we can défine equivalence relation. On le X/R comme le espace topologique quotient.

Definition

A topological space X

With an equivalence relation $\sim$, we can construct a new topological space $X/\sim$ from another topological space X called the *quotient space*. So we divide X in classes which are the elements of the new space. we will see for example the sphere S^n is quotient space : $D^n/\partial D^n$. One can verify that makes a *partition* of X.

Remarks:

−On remarque que l'espace X/R n'est pas nécessairement séparé même si X est séparé ⏏

◯ ***Induced topology:***

Let E be a set and $A \subset E$ in we define a topology

Definition:

$A \subset E$ is if $\tau_A = \{: A \cap E\}$. Is a topology.

Proof:

ii)let $A_1, \dots, A_1$
iii)

The induced topology is known as weak or subspace topology .

Normal Topology:

Definition:

we say that the topological space $(E, \boldsymbol{\tau})$ is *normal* if for every disjoint A and B there neighborhoods respectively such that

$$v_1 \cap Av_2 = \emptyset$$

Every two disjoint closed sets of have disjoint open neighborhoods. We see that all metric spaces are normal.

We here a famous lemma due to the Russian mathematician Pavel **Ureshon**[6]$(1898 - 1924)$. This lemma gives necessary and sufficient conditions to a topological space to be normal.

Ureshon lemma (192) :

a topological space $(E, \boldsymbol{\tau})$ is *normale* iff

◯ *Continuity:*

In this new language of point-set topology, the continuity of a function in Analysis becomes :

[6] **Ureshon** died young drown in France, hem de work on topology we have seen that with Alexdrove they both compactness.

Definition:

A function $f:(E,\tau_1)\longrightarrow(F,\tau_2)$ is continuous if

$$f^{-1}(A\in\tau_2)\in\tau_1$$

another is : f is continuous if for every x in E and every neighborhood A of $f(x)$, $f^{-1}(A)$ is neighborhood of x in E.

A good question is : what is the weakest topology that makes f continuous? We will see the answer to this next.

We *connectedness* or *connectivity* We say C is connected if it is $C=A\cup B$ with $A\cap B=\emptyset$. And we this introuded the symbol in place of $\cup$

$$A\coprod B$$

Which is the disjoint is there a continuous $f:\mathbb{R}\longrightarrow S^0$. Sometimes a function f is continuous on $A\subset E$ can it conntouson the whole E?

Tietze's theorem

Every continuous f can be extended continuously to the whole space.

This leads us to a *restriction* of f at $A\subset E$. I is a connected set.

Remarks:

–Topology is interested also when if f^{-1} also continuous, in this case f is called *homeomorphism*. Homeomorphisms are studied in algebraic topology. One of the results is that the intervals $[a,b]$ and $]a,b[$ are not homeomorphic ⏏

The theorem has a relation with a lemma proved by young Pavel **Urysohn**$(1898-1924)$ in . this is link to *normal* spaces.

◯ ***Weak topology***:

We can prove that if $\tau_2\subset\tau_1$ (τ_1 finer than τ_2) then $id_X:(E,\tau_1)\to(E,\tau_2)$ is continuous. Let the map $f:E\to F$ We seek here the *weakest* topology (which means the coarsest) that makes all the map continuous.

◯*Topological Dimension theory*

See James:.

Mathematicians love generalization to discover similarities mathematicians have tried to generalize the concept of *dimension* aux abstract topological spaces. We here the methods used to generalize the concept of dimension from $\mathbb{R}^n$ to a general topological space. As in a vectorial space V, a dimension which is the maximum number of linearly Independent vectors in V..

Lebesgue definiton

Ureshon

Covering dimension

Metric dimension

Hausdaurf dimension

Exercises

1) show that $\emptyset \subset \emptyset$
1) Give $\mathcal{P}(\emptyset)$.
1) show that if : $A \subset \emptyset$ then $A = \emptyset$.
1) Show that $X \setminus (X \setminus Y) = X \cap \bar{Y}$.
1) Show that $A\Delta B = (A\backslash B) \cup (B\backslash A)$
1) Show that if $X \subset Y$ then $X \setminus Y = \emptyset$.
1) Show that $(X \setminus Y) \cap Y = \emptyset$.
1) let subsets of E. Show that :
- $E \setminus \bigcup_1^n A_i = \bigcap_1^n (E \setminus A_i)$
- $E \setminus \bigcap_1^n A_i = \bigcup_1^n (E \setminus A_i)$
1) Show that $X \cap (Y - Z) = (X \cap Y) - (X \cap Z)$.
1) Show that $A = \emptyset \;\; iff \;\; A\Delta B = B$.
1) Show that $(A\Delta B)\Delta(A \cap B) = A \cup B$.
1) Show that $(A\Delta(A \cap B) = A - B$.
1) What is the power set of $\emptyset$?
1) Let $f : E \longrightarrow F$ a map show that :
- $f(A \cup B) = f(A) \cup f(B)$.
1) show that $f(x) = x$ is injective. Is $f(x) = x^2$ injective ?
1) show that every part of a countable set is countable.
1) show that the intersection of countable sets is countable.
1) show that the sets of odd and even numbers are countable
1) show that: $\mathbb{Z}$, $\mathbb{Q}$ et $\mathbb{N}^2$ are countable. Show that there are infinitely many cardinals.
1) Show that one can extract a ensemble countable set from an infinite set. Show that one can extract a non-countable set from a non-countable set.
1) show that the union of countable set is countable.
1) show that every set is equipotent to itself.
1) what is $\bigcap_1^\infty]-1, \frac{1}{n}[$, $\bigcap_1^\infty]-\frac{1}{n}, \frac{1}{n}[$, $\bigcap_1^\infty]-n, n[$?

1) what is the difference between $\bigcup_1^{\infty}[\frac{1}{n},1]$ and $\bigcup_1^{\infty}]\frac{1}{n},1]$?

1) knowing that $\sum_{i\in\mathbb{N}^*}\frac{1}{i^2}=\frac{\pi^2}{6}$ give the sum of the series : $\sum_{i\in\mathbb{Z}^*}\frac{1}{i^2}$; $\sum_{i\in\mathbb{Q}^*}\frac{1}{i^2}$

1) Find : $\bigcup_{n=1}^{\infty}[-\frac{1}{n},\frac{1}{n}]$, $\bigcup_{n=0}^{\infty}]-n,n[$, $\bigcap_0^{\infty}]-n,n[$, $\bigcap_1^{\infty}[-\frac{1}{n},\frac{1}{n}]$

1) Let $E=\{0,1\}$. Give all the topological spaces over E.

1) show that a topology is a π −system.

1) Let τ_A be an induced topology. Show that $E,\emptyset\in\tau_A$

1) is the set $\{\emptyset,\mathbb{R},all\ ]a,b[\}$ a topological space over $\mathbb{R}$?

1) is the set $\{\emptyset,\mathbb{R},all\ ]-a,a[\}$ a topological space over $\mathbb{R}$?

1) is the set $\{\emptyset,\mathbb{R},all\ ]a,+\infty[\}$ a topological space over $\mathbb{R}$? What is the interior of [0,1]?

1) how is 0 a limit point to the set $E=]0,1]$?

1) is 0 a limit point to the set $E=\{\frac{1}{n}\}$?

1) let $E=\{0\}\cup[1,2]$, does $0\in E'$? What is it ?

1) When is E closed ?closure of a set $\bar{E}$.

1) When is $E\subset E'$? When is $E'\subset E$?

1) Show that if $E\subset E'$ then $E=E'$.

1) let $\mathcal{P}$ a partition of $E(\cap=\emptyset)$ is $(E,\emptyset,\mathcal{P})$ a topological space? Equivalence relation on E defines a partition of E.

1) show that the discrete space is **Hausdorff**(separated or T_2).

1) when is $f:E\longrightarrow F$ continuous ? the inverse set function is much more well-behaved.

1) give a homeomorphism between two sets of $\mathbb{R}$.

1) define a dense set in $\mathbb{R}$. Explain $\overline{\mathbb{Q}}=\mathbb{R}$. Is $[0,1]$ dense in $\mathbb{R}$.

1) let (E, $\boldsymbol{\tau}$) a topological space, find the derived set of the following sets:

i) $A=[0,2[$

ii) $B=]-1,1[\cup[2,3]$

1) Find the adherent points of

i) $A=[0,2[.$,

ii) $B=]0,2[.$

1) show that $A^o\subset A$.

1) give accumulation points of the sets : $\mathbb{N}$,]0,1], $\mathbb{R}/\mathbb{Q}$.

1) show that A is open if $int(A)=A$.

1)show that if x is isolated then $\{x\}$ is open

1) montrer que $E\subset\bar{E}$

1) let E a discrete topological space, show that tout point $x\in E$ is isolated.

1) in (R,) is the set $\{1,\ \frac{1}{100},\ \frac{1}{200},\dots\}$ closed, why?

1) what is the derived set (definition) of : $A=]0,1]$.

1) what are the accumulation point**s** of $E=\{\frac{1}{n}\}$? Is $\frac{1}{3}\in E'$?

1) Let $A=]0,1[\cup\{2\}$. What is 2 for A?

1) is a circle the perimeter of a disc?

1) show that every discrete topological space is separated.

1) let $\boldsymbol{\tau}$ indiscrete topology over E, show that every point of E is limit of a function.

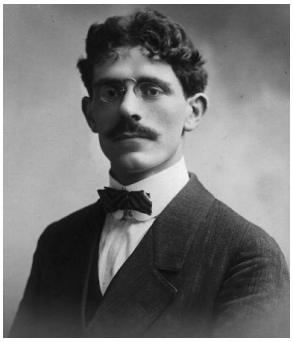
1) let (E, $\boldsymbol{\tau}$) show that $\boldsymbol{\tau}$ is topological basis.

1) soit $E=\{a,b,c\}$ donner tous les topologique over E. make a comparison between them.

1) soit E et $\mathcal{P}$ une partition de E, est-ce que(E, $\mathcal{P}$) est un espace topologique ?.
1) soit ($\mathbb{R}$) espace topologique expliquer why {0} is not open. Is it close ?
1) shwo that the sets $\mathbb{Z}$, $\mathbb{Q}$ are closed.
1) soit $E = [-1,1]$ donner un topologie sur E.
1) show that the set $E = [-1,1[\cup]\, 1,2]$ is closed.
1) show by using covers that $\mathbb{R}$ is not compact.
1) Let E ={a, b, c} , give all the topological spaces on E. Expliquer pourquoi {a} est un ouvert.
1) Explain why $]-\infty,0] \cup [0,+\infty[$ and $[0,+\infty[$ are closed sets of $(\mathbb{R},|\quad|)$.
1) montrer que l'intersection d'espaces topologiques est un espace topologique.
1) montrer que l'union d'espaces topologiques n'est pas nécessairement un espace topologique.
1) why is {0} an open for pour $E = \{0,1\}$ but not for $(\mathbb{R},)$)
1) montrer que E et ∅ sont des fermés dans un espace topologique.
1) montrer qu'une topologie contient tous les singletons alors elle est discrète.
1) montrer qu'une réunion finie de fermés est un fermé
1) show that the any intersection of closed is closed.
1) Let f a continuos function, show that the image de ouvert par f n'est pas nécessairement un ouvert. Same thing of r a closed set.
1) tell why in a topology τ on a set E, ∩ is finite.
1) show that ∅ is a neighborhood for every x.
1) show that $\tau_1 \cap \tau_2$ is a topology , what about $\tau_1 \cup \tau_2$?
1) let $E = \{x\}$, give all possible topologies on E? Are they Hausdaurf?
1) show that an indiscrete topology having two or more elements cannot be separated. Show that it is not metrizable.
1) let E a set and P a partition of E, is $\{E, \emptyset, P\}$ a topological space ?
1) Show that $\mathbb{N}$ and $\mathbb{Z}$ are closed in $\mathbb{R}$.
1) let $f(x) = x^2$. give $f(\emptyset)$ et $f^{-1}(\emptyset)$.
1) montrer que $f(x) = x + 1$ est continue au point $x = 0$
1) trouver $f(]-1,2])$ pour $f(x) = x^2$. $f^{-1}(]-1,2])$ Est que f^{-1} existe ?
1) Show that $f(x) = constante$ is continuous.
1) let $f: E \longrightarrow F$. Find $f^{-1}(\emptyset)$ and $f^{-1}(F)$.
1) show that if $\tau_2 \subset \tau_1$ (τ_1 finer than τ_2) then $id_X : (E, \tau_1) \to (E, \tau_2)$ is continuous.

Metric Spaces

We have observed that in a topological space, the relationship between the elements of a set E is very vague, stemming from the fact that the structure of a topological space is very simple. We can delve deeper into our structure and add a distance between the elements to better locate their points, which leads us to metric spaces. A metric space is more specialized than a topological space; every metric space is a topological space, but not the other way around. The notions of proximity and proximity are more clearly defined… alongside continuity, we can discuss the concepts of convergence and limits. In our new topology, we will see a stronger structure not only neighborhood, but distance between points.



Fréchet

◯ Metric on a Set:

Let E be a set, we say that $d : (E, E) \longrightarrow \mathbb{R}^+$ is a *distance* function on E if :

Definition :

$-\forall x, y \in E : d(x, y) = 0 \iff x = y$ (Separation or reflexivity).

$- d(x, y) = d(y, x)$ (Symmetry)

$-d(x, y) \leq d(x, z) + d(z, y)$ (triangular inequality)

Remark :

–The separation means x and y are separated since $x \neq y$.

–The distance d is also called a *metric*, appellation due to French Maurice **Fréchet**$(1878 - 1973)$ in his PhD dissertation[7] (1906) ⏏

Examples :

– The most known of the reader because distance Euclidian to Euclid and his usual geometry.the usual distance over $\mathbb{R}$ is

$$d(x, y) = |y - x|$$

[7] Under the title : "*On Some Point of Functional Calculus*". In it he defined the structure of *metric space*.

It is the distance between x and y on the real line. on the plan $\mathbb{R}^2$ it is

$$d(x,y) = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

in the space $\mathbb{R}^3$ it is

$$d(x,y) = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2 + (z_2 - z_1)^2}$$

They are distances of Euclid geometry they are called *Euclidian distance* or *usual distance* which may generalized to $\mathbb{R}^n$

$$d(x,y) = \sqrt{\sum_{k=1}^{n} (x_i - y_i)^2}$$

And the usual is just a special case as $\sqrt{(y-x)^2} = |y - x|$.

– The *discrete* or trivial distance

$$d(x,y) = \begin{cases} 1 \ si \ x \neq y \\ 0 \ si \ x = y \end{cases}$$

which has no natural meaning. Another is a pseudo metric called the *indiscrete* metric $d(x,y) = 0$ for every x, y it is associated to the indiscrete topology.

– Fréchet distace: introduced in his PhD thesis (1906) **Fréchet** gave the following distance

$$d(x,y) = \sum_{i=1}^{i=n} \frac{h}{}$$

défini des nouvelle, la norme les espaces abstraits. Les mathématiciens to their imagination have given distances.

– Distance of uniforme converence :Soit (E,d) un espace métrique la est de la comme suit

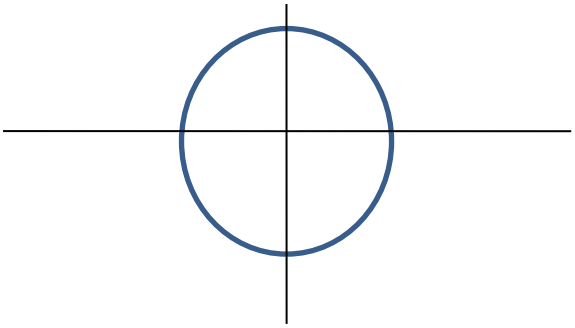
$$(f,g) = sup_x d_u(f,g)$$

– <u>distance entre deux ensembles</u> :let (E, d) un espace métrique la est de la comme suit

$$(f, g) = sup_x d_u(f, g)$$

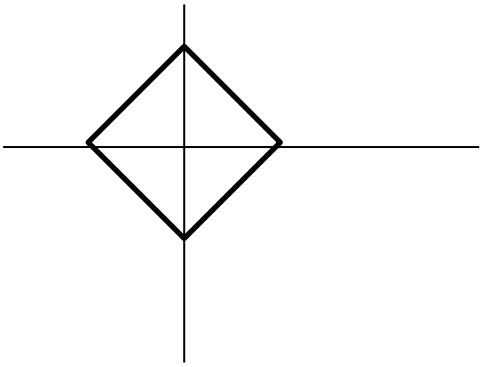
1) distance $d_2(x, y)$

$$d(x, y) = \sum_{i=1}^{i=n} |x - y|$$



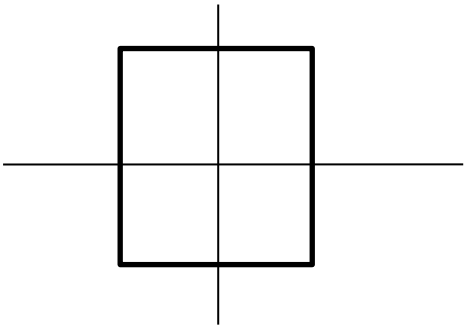
2) distance

$$d(x, y) = \sum |a_i|$$



3) distance $d_\infty(x, y)$ define

$$d_\infty(x, y) = max(|x_i|)$$



The comes from roughly the norm :

$$\|x\|_p = (\sum_{k=1}^{\infty} |x_k|^p)^{1/p} \quad ;$$

When $p \longrightarrow +\infty$.

If the mas so we have a square. And we can now make a comparison

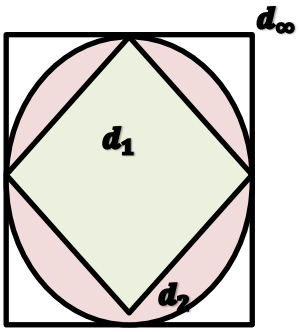


Fig. between the three

Three be sure that are true.

◯ *Metric Space:*

We have now a distance we define a new structure:

Definition:

let E be a non-empty set, and d a distance over E , then (E, d) is a new structure called a *metric space*.

Remark :

–The problem of when a topological space is metrisable is difficult ⏏

Examples :

– We have seen that $\mathbb{R}$ is a topological space, it is also a metric space ⏏

Let E be a metric space, a set A is open in E if for every $y \in A$, $\exists \varepsilon > 0$ such that for any $x \in E$ we have $d(y,x) < \varepsilon$ belongs to A.

Theorem

A set with a metric is a metric space(E,d). Every metric space is **Hausdaurf**.

We say that the topology of(E,d) is induced by the metric d. this result that a metric space is necessarily separated

So a metric space is necessarily separable. This means that a space which is not Hausdaurf cannot be metric.

We have metric every metric a topology, we can find topologies that are not *metrizable*.

◯ *Balls:*

in a topological space we cannot speak of *bounded* part , but in a metric space this notion exists thanks to the notion of distance:

Definition:

let (E, d) be a metric space, we say that A *bounded* if there exists a closed ball $B(x_0,r)$ tel que A $\subset B(x_0,r)$.

the balls in metric spaces play the same role as the role of neighborhoods in topological spaces not metrizable. An open set in (E,d) is every part $\mathcal{O}$ of E such that

$$\forall x \in \mathcal{O}\ , \exists r > 0, B(x,r) \subset \mathcal{O}$$

Balls are a generalization of *disk*. In the plane we talk of disks. When on cette distance on peut les point with closed balls et open in a set E.
A subset A is open in a metric space (E,d) if for every $x_0 \in A$ there exists $\varepsilon > 0$ such that for every $x \in E$ verfying $d(x_0,x) < \varepsilon$ is in A.
The *open ball* of center x_0 et de rayon r is defined like the set

$$B(x_0,r) = \{y \in E, d(x_0,y) < r\}$$

the *closed ball* of centre x et de rayon r est

$$B(x_0, r) = \{y \in E, d(x_0, y) \leq r\}$$

In $\mathbb{R}$ with the usual distance les balls are the intervals. Dans $\mathbb{R}^2$ sont les disques, dans $\mathbb{R}^3$ les boules sont les spheres. On va voir $\mathbb{R}^2$ trois de boules qui diffère de formes selon la distance.

On donne ici l'équivalent dans une espace métrique du théorème de Bolzano Weierstrass

Theorem:

De toute suite borne peut extraite une sous suite .

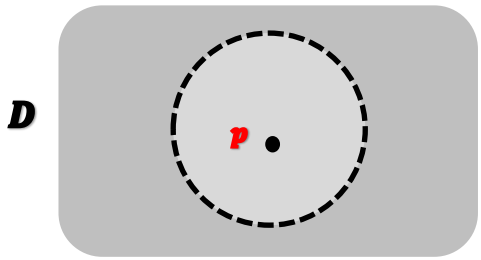


Fig. $\boldsymbol{D}$ is a neighborhood of p

Remarks:

There are interesting topologies that are not metrizable:

- Sierpinski topology.

- the weak* topology.

- Zariski topology.

- the topology of the space test functions $\mathcal{D}$ in the theory of the distributions.

- topology of simple convergence⏏

◯ *Continuity in a Metric Space:*

Continuity is a central subject of study in topology as **Dixmier** says :

Topology is the study of notions of limit and continuity and thus is, in principle, very ancient. One of the sources of topology is the efforts of clarify the theory of real-valued functions of a real variable: uniform continuity, uniform convergence, equicontinuity, Bolzano-Weierstrass theorem. Cauchy was one of the pioneers in this direction, but he errors that slip into his work prove how hard it was

in a metric space (E, d) becomes :

Definition :

Let $f : (E, d_1) \longrightarrow (F, d_1)$, f is continuous if

$$\forall\, \varepsilon > 0\,, \exists \delta 0 :\ d(x, y) < \delta \implies d(f(x), f(y)) < \varepsilon$$

Ce qui l'inverse est un ouvert. A function f(x)) is continuous if

$$\forall\, \varepsilon > 0\,, \exists :\ |x - x| < \delta \implies |(f(x), f(y))| < \varepsilon$$

In analysis, there is between continuity and uniform continuity.

Definition :

Soit f :(E,d_1)⟶(F, d_1) est *uniformément* continue si f()

$$\forall\, \varepsilon > 0\,, d(x, y)\ d(f(x, f(y))$$

we have f is Lipchitz

$$\forall\, \varepsilon > 0\,, f(x - x) \leq c$$

and we have

(f Lipchitzian) $\implies$ (f uniformly continuous) $\implies$(f continuous)

The inverse is not always true but we have the result :

Proposition:

if f is continuous on a segment $[a,b]$ the nit is uniformly continuous over $[a,b]$.

Dfvv

Exercises

1) show that any metric verifies $d(x,y) \geq 0$.
1) show that : $|d(x,y) + d(x,z)| \leq d(y,z)$
1) show that : $|d(x,y) - d(x,z)| \leq d(y,z)$
1) let d be a distance show that : $d_1(y,z) = \frac{d(y,z)}{1+d(y,z)}$ is a distance.
1) Show that $\{E, \emptyset, \{a\}\}$ is not metrizable.
1) show that the following are metrics :
$-d_\infty(x,y) = max|x-y|$
$-d(x,y) = \sum_{i=1}^{i=n}|x-y|$
$-d(x,y) = |x-y|$
$-d(x,y) = |x_1 - y_1| + |x_2 - y_2|$
$-d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$
1) show that $min(x,y) + max(x,y) = x + y$.
1) show that if d is a metric $a \bullet (d)$ is also a metric for $a > 0$.
1) show that the following are not metrics :
$-d(x,y) = (x-y)^2$
$-d(x,y) = \min\{|x_i - y_i|\}, 1 \leq i \leq n.$
1)show that the discrete distance is a metric.

Normed Spaces

in our survey on the science topology , we have seen that in a topological space E there is not concepts like bounded part, distance, balls... then we have added more structure to our edifice to get metric spaces, spaces richer where we can define a distance between elements of E , then we lack some on a enriched to find les best structure which are the normed spaces.

Banach

◯ *Norm on a Set:*

For a metric space to be a normed space, it has to be a vector space first . First we define a norm. Let E be a set. We say that $\|.\| : (E,E) \longrightarrow \mathbb{R}^+$ is a *norm* over E

Definition :

$i)$ $\|\lambda \bullet x\| = |\lambda| \bullet \|x\|$ $\lambda \in \mathbb{R}$ (homogeneity)

$ii)$ $\|x\| = 0 \Leftrightarrow x = 0$ ($\|.\|$ is not degenerated)

$iii)$ $\|x + y\| \leq \|x\| + \|y\|$ (triangular inequality)

Remarks:

–in this course, a norm is defined only on vector spaces.

–The distance between two elements associated to any norm $\|\ \|$ is defined as $d(x,y) = \|x - y\|$ ⏏

Examples

1) $d(x,y) = |x - y|$ between two real number is a norm on $\mathbb{R}$. it verifies the three axioms or conditions i), ii) and iii).

2)

3) ⏏

Proposition :

Every norm over E is a distance over E.

The inverse is false, there are distances that are not norms. Alors pour que distance une norme on a les conditions

Semi-norms:

If the condition is not fulfilled the norm

$i) \|\lambda.x\| = |\lambda|.\|x\|$ (homogénéity)

$ii)\ \|x\| = 0 \Longrightarrow x = 0$

$iii) \|x + y\| \leq \|x\| + \|y\|$

Remarks:

−every norm is a seminorm. ⏏

Examples

1) open ball $\|x - y\| < \varepsilon$
1) closed ball $\|x - y\| \leq \varepsilon$⏏

Equivalent norms:

when are $\|\ \ \|_1$ and $\|\ \ \|_2$ equivalent ? For dimension n finite ?

Definition:

E définie a norm $\|.\|$, we say that $(E, \|.\|)$ is a normed space.

Remarks:

−Th set e $(\mathbb{R}, |.|)$ is a topological, metric and normed space.

The completeness is treated as in the metric space. ⏏

Normed vector spaces

Normed spaces were defined by Frigyes **Riesz**$(18-19)$ en 1906. The normed vector spaces are the most important of all. Vector spaces are easy to work with so they have been extensively studied and a complete theory.

◯ *Famous inequalities :*

There are many inequalities Very important have been used by mathematicians that enable us to compare between norms.

Minkowski:
Let x and $y \in E$

$$\left(\sum |x_i + y|^p\right)^{1/p} \leq \left(\sum |x_i|^p\right)^{1/p} + \left(\sum |y|^p\right)^{1/p}$$

It is the Phytogorian inequality is a special case

$$|x + y| \leq |x| + |y|$$

It is just when $n = 2$ and $p = 1$.

Holder:
Let x and y

$$\sum_{i=1}^{n} |x_i \bullet y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \bullet \left(\sum_{i=1}^{n} |y|^p\right)^{1/p}$$

A particular case is when $p = q = 2$, we find the Cauchy–Schwartz inequality encore also called Bouniakowsky[8] inequality.

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \bullet \left(\sum_{i=1}^{n} |y|^2\right)^{1/2}$$

[8] after Ukrainian mathematician Victor **Bunyakovski**$(1804-1889)$ who made contributions in mechanics and number theory.

we can go further in our structure, et on find metric spaces to normed spaces. The most important are the normed vectorial spaces very used in functional analysis. The most interesting are the normed spaces of functions called *functional spaces*, very important in functional analysis. On va alors des exemples très importants des espaces : les espaces de Banach.

◯ *Banach Spaces :*

See now a very general type of spaces used extensively in mathematics. They are normed space .

Definition :

A Banach space is a complet normed vectorial space.

Remarks:

—A Banach space is called also *polish*, to Polish school in XX century. So i.e. polonais complet. Banach spaces are named after polish mathematician Stephan **Banach**$(18-19)$ in his book in published in Frenh language : *Operateur Linaire*(194) , he nommed his spce *"space B"* . the letter B honor of French mathematician **Baire**$(18-19)$ puis par Fréchet. Banach spaces are very used in functional analysis ⏏

Examples

—$\mathbb{R}$ is a Banach space with the usual norm ⏏

Weak topology

We know that topology with continuity. So if $f : E \longrightarrow F$ is continuous then all $f^{-1}(\mathcal{O}_i \in \tau_2)$ should be open in E. So we look for the topology that is sufficient for f to be continuous. For this, we met usually strong topology and weak topology. We say that a topology is *weak*. If it contains more open sets

In Swedish Per **Enflo**$(1944-)$ showed that that Banach spaces have basis

◯ *Hilbert Spaces :*

The some useful geometric properties like orthogonality

Scalar Product :

We define now a relation that will give us a special norm very important later.

Definition :

We say scalar product is a map overr $E \times E \longrightarrow \mathbb{R}^+$.

$$x * y = \sum_{i=1}^{n} x_i \bullet y_i$$

Remarks:

–The is also dot product. The appellation *scalar* to vector product. There is another called *dot product* which is. ⏏

–We know from geometry that two vectors in $\mathbb{R}^2$ are *orthogonal* $v_1 \perp v_2$ if

$$v_1 * v_2 = 0$$

⏏

We want to generalize to other spaces

Definition :

A space is *orthogonal*

$$\forall\, x, y \in E : \; x \perp y$$

A Hilbert space, noted[9] $\mathbb{H}$, is a normed vector space from scalar product. We see now an important real Hilbert space

Definition :

in a Hilbert space the *parallelogram equality*

$$\forall \text{ x, y} \in \mathbb{H}: \; \|x - y\|^2 + \|x + y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Examples

[9] This notation nis c shard by hyperbolic space $\mathbb{H}^2$ with 2, and the skew field of quaternions $\mathbb{H}$.

−$\mathbb{R}$ is a Banach space with the usual scalar product.

Inner Product:

The difference between inner and scalar product is that inner is a generalization of dot product to functions. This especially when the space is infinite dimension.

Definition :

inner product is over $E \times E$.

$$\langle u, v\rangle = \int_a^b uvdx$$

We find as a special case $\langle u * v\rangle = uv = \sum_{i=1}^{n} x_i \bullet y_i$ have proprieties:

$-\langle x+, y\rangle =$
$-\langle x+, y\rangle =$
$-\langle x * y\rangle =$

Euclidian spaces:

This is an important metric space, Euclidian

Definition

An Euclidian space is a finite dimensional vector space over the reals equiped with a dot product.

It is of $(x_1, \dots, x_n)$. The Euclidian space is sometimes noted E^n the Euclidian space is $\mathbb{R}^n$

Remarks:

−The appellation "Hilbert space" is due to Hungarian mathematician Von **Neumann**$(18 - 195)$ a student of **Hilbert** at the University of Göttingen.

−Often a normed space with scalar product is called *pre-Hilbertian* , in the finite case every prehilbertian space is a Hilbert space. In the infinite case it is not true ⏏

Examples

−$\mathbb{R}^n$ is an Euclidian space. ⏏

◯ *Functional Spaces :*

Functional spaces are spaces of functions used in Functional Analysis related to differential and integral equations. They are vector(or linear) spaces of *infinite dimension*

1) the continuous function over, noted $C^0(\quad)$ muni the norm

$$\|f\| = \sup$$

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$$\|f\| = \sup$$

1) the continuous function over, noted $C^0(\quad)$ muni the norm

$$\|f\| = \sup$$

in functional analysis , we know that the space $C^0([0,1],)$ is not complet for the norm $\int_0^1 f dx$ne for a sequnce of continuoes function may converges to a discontinue function.

Exercises

1) show how every norm is a metric.
1) show that the discrete metric is not a norm.
1) show that any metric given by a norm should verify:
$i)\ d(\alpha x, \alpha y) = |\alpha| d(x,y).$
$ii)\ d(x+\alpha, y+\alpha) = d(x,y).$
1) show that $\|0\| = 0.$

1) show that $\|x\|_p = (\sum_1^n |x_i|^p)^{1/p}$ is a norm on $\mathbb{C}$. It is the p-norm.
1) show that the following are norms :
−set of sequences $x = \{x_k\}$

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} \quad ;$$

$-\|x\|_{C[0,1]} = \max|f(t)|$(set of continuous functions over $[0,1]$)
1) Let S the set of sequences a and b

$$d(x,y) = \sum \frac{1}{2^n} \times \frac{|y_n - x_n|}{1 + |y_n - x_n|}$$

Show that it is a distance. Is it a norm?

1) second triangular inequality : $\|x - y\| \geq |\|x\| - \|y\||$.

1) Show that C^K[a,b] is a linear variety in C^n [a,b] $(k > n \geq 0)$

1) let E and F be two vector spaces with norm $\|x\|$ and $\|y\|$ respectively, show that $\|(x,y)\|$=$\|x\|$ + $\|y\|$ is a norm over the Cartesian product $E \times F$.

1) show that the space $\mathbb{R}\backslash\{0\}$ is not complet.

1) show that every normed space is separated.

1) show that the triangular is a particular case of Minkowski.

1) Show that the space $E =]0,1]$ is not Banach.

1) show that if a linear f is continuous at 0 then it's continuous at every x.

1) show that : $\|A^n\| \leq \|A\|^n$.

1) Draw a parallelogram and prove *the parallelogram inequality*

$$\forall\, x, y \in \mathbb{R}^2 : \|x - y\|^2 + \|x + y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

1) show that toute norme sur E est une distance

1) montrer que tout espace normé est séparé

1) show that the following are norms :

$d_\infty(x,y) = max|x - y|$
$d(x,y) = \sum_{i=1}^{i=n} |x - y|$
$d(x,y) = |x - y|$
$d(x,y) = |x_1 - y_1| + |x_2 - y_2|$
$d(x,y) = \max\{|x_1 - y_1|\,, |x_2 - y_2|\ \}$

1) show that the inverse of a linear operator A if it exists is linear.